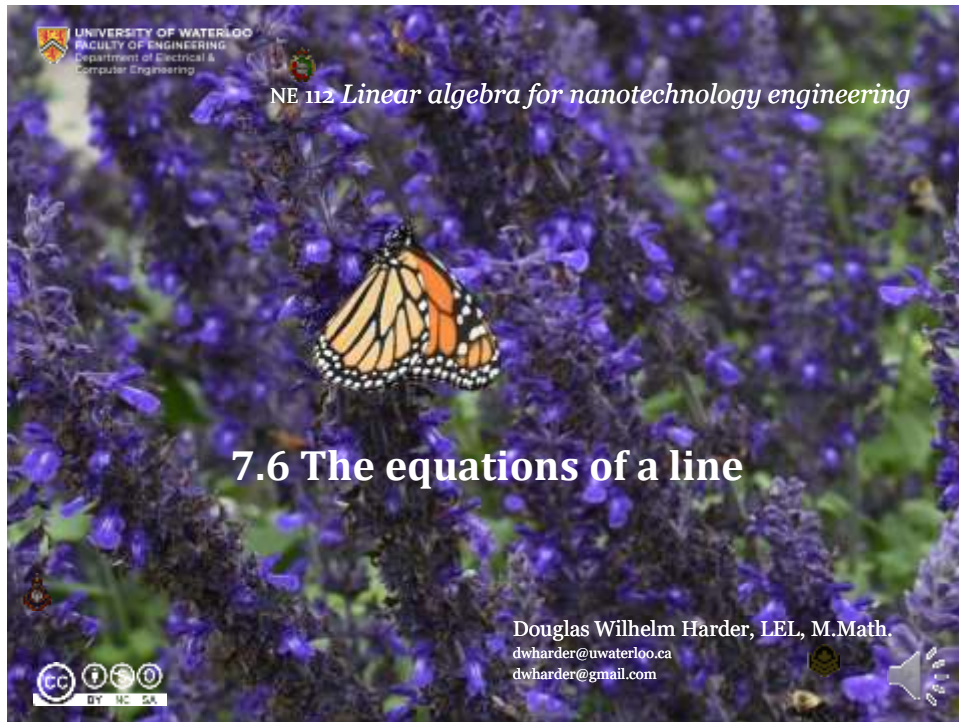




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The equations of a line

Introduction

- In this topic, we will
 - Review the equations of a line
 - Discuss how we can solve the system of linear equations to find a vector representation of that line
 - Describe how we can use the cross product and the Moore-Penrose pseudoinverse to algorithmically find that vector representation
 - Discuss the differences between algorithmic and algebraic solutions to problems

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The equations of a line

Equation of a plane

- In $\mathbb{R}^3$, a plane is represented algebraically by a single equation:

$$ax + by + cz + d = 0$$
 - You of course are familiar with this:
a single linear equation in three unknowns

$$ax + by + cz = -d$$
 - Therefore, there are generally two free variables
 - It is these two free variables that define a plane
 - There is only one circumstance where there are three free variables and infinitely many solutions:

$$0x + 0y + 0z + 0 = 0$$
 - The only time when there are no solutions is when t is non-zero in this equation:

$$0x + 0y + 0z + d = 0$$

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The equations of a line

Equations of a line

- In $\mathbb{R}^3$, a line is represented by a system of two linear equations:

$$a_1x + b_1y + c_1z + d_1 = 0$$

$$a_2x + b_2y + c_2z + d_2 = 0$$
 - We will assume at least one coefficient of x, y or z is non-zero for each equation and ignore the pathological cases
 - The line is the intersection of the two planes
 - Two different planes, if they intersect, will intersect along a line
 - Thus, our augmented matrix is:

$$\left(\begin{array}{ccc|c} a_1 & b_1 & c_1 & -d_1 \\ a_2 & b_2 & c_2 & -d_2 \end{array} \right)$$

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The equations of a line

Parallel or equal planes

- From secondary school, you are aware that there are two possibilities:
 - The vectors $\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$ are linearly dependent,
 - so one is a multiple of the other
 - Thus, the row-equivalent matrix in row-echelon form is

$$\left(\begin{array}{ccc|c} a_1 & b_1 & c_1 & -d_1 \\ 0 & 0 & 0 & -d_2 + \frac{a_2}{a_1}d_1 \end{array} \right)$$
 - If $-d_2 + \frac{a_2}{a_1}d_1 = 0$, there are two free variables and the two equations describe the same plane
 - Otherwise, they describe two parallel planes, so no solutions
 - If $a_1 = 0$, we go to Column 2 or 3, but the result is the same

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The equations of a line

Non-parallel planes

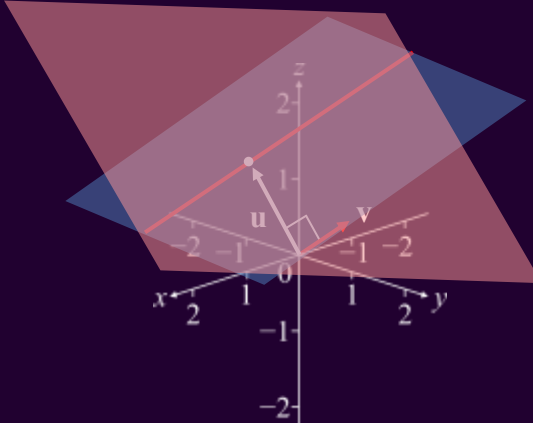
- From secondary school, you are aware that there are two possibilities:
 - Alternatively, the vectors $\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$ are linearly independent,
 - so there is only one free variable
 - This describes a line

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The equations of a line

Ideal vector representation

- Here we see two non-parallel planes
 - We want to find the vector representation of this line



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The equations of a line

Vector representation of a line

- You could solve the system of linear equations
 - For example, given two planes:

$$4x + 2.4y - 1.2z - 4.6 = 0$$

$$2x + 5.2y + 4.8z - 5.5 = 0$$

$$\begin{pmatrix} 4 & 2.4 & -1.2 & | & 4.6 \\ 2 & 5.2 & 4.8 & | & 5.5 \end{pmatrix} \quad \begin{pmatrix} 0.67 \\ 0.8 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1.11 \\ -1.35 \\ 1 \end{pmatrix}$$
 - This is $\mathbf{u} + \alpha \mathbf{v}$
 - We now need to normalize $\mathbf{v}$ and make $\mathbf{u}$ orthogonal to $\mathbf{v}$
 - Problem: this is not easily programmable...

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The equations of a line

Vector representation of a line

- Instead, start with the vectors perpendicular to the planes
 - First, let $\mathbf{v}_1 \leftarrow \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}$ and $\mathbf{v}_2 \leftarrow \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$
 - We now assign $\hat{\mathbf{v}} \leftarrow \frac{\mathbf{v}_1 \times \mathbf{v}_2}{\|\mathbf{v}_1 \times \mathbf{v}_2\|_2}$
 - This must be parallel to the line
 - We now need $\mathbf{u}_\perp$

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The equations of a line

Vector representation of a line

- There is a fascinating construct called the *Moore-Penrose pseudoinverse*

$$A^+ = \frac{\begin{pmatrix} a_1(b_2^2 + c_2^2) - a_2(b_1b_2 + c_1c_2) & a_2(b_1^2 + c_1^2) - a_1(b_1b_2 + c_1c_2) \\ b_1(c_2^2 + a_2^2) - b_2(c_1c_2 + a_1a_2) & b_2(c_1^2 + a_1^2) - b_1(c_1c_2 + a_1a_2) \\ c_1(a_2^2 + b_2^2) - c_2(a_1a_2 + b_1b_2) & c_2(a_1^2 + b_1^2) - c_1(a_1a_2 + b_1b_2) \end{pmatrix}}{(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2}$$
 - It turns out that $\mathbf{u}_\perp \leftarrow A^+ \begin{pmatrix} -d_1 \\ -d_2 \end{pmatrix}$

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The equations of a line

Vector representation of a line

- For the example we just saw:
 - First, let $\mathbf{v}_1 \leftarrow \begin{pmatrix} 4 \\ 2.4 \\ -1.2 \end{pmatrix}$ and $\mathbf{v}_2 \leftarrow \begin{pmatrix} 2 \\ 5.2 \\ 4.8 \end{pmatrix}$
 - We now assign $\hat{\mathbf{v}} \leftarrow \frac{\mathbf{v}_1 \times \mathbf{v}_2}{\|\mathbf{v}_1 \times \mathbf{v}_2\|_2} = \begin{pmatrix} 0.551250467470862 \\ -0.670439757734833 \\ 0.496622042766543 \end{pmatrix}$
 - Finally,

$$\mathbf{u}_\perp \leftarrow A^+ \begin{pmatrix} -d_1 \\ -d_2 \end{pmatrix} = \frac{\begin{pmatrix} 186.88 & -12.48 \\ 53.248 & 85.312 \\ -135.552 & 129.024 \end{pmatrix}}{1037.9776} \begin{pmatrix} 4.6 \\ 5.5 \end{pmatrix} = \begin{pmatrix} 0.7620665417057169 \\ 0.6880271790065605 \\ 0.0829428303655108 \end{pmatrix}$$

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The equations of a line

Algorithmic versus algebraic solutions

- In this course on linear algebra, we focus a lot on algebra
 - However, when implementing solutions, it is often best to find efficient algorithms that solve the same problem
- If you were to look at the matrix A^+ and asked to program it, you likely can
 - Later in this course, we may be able to derive that matrix

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The equations of a line

Summary

- Following this topic, you now
 - Understand the equations of a line
 - Know how to find a vector representation of that line by solving a system of linear equations with one free variable
 - Understand that there are more algorithmic approaches to solving the same problem
 - Are aware that algorithmic approaches can be more easily programmed than algebraic approaches

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The equations of a line

References

[1] [https://en.wikipedia.org/wiki/Line_\(geometry\)](https://en.wikipedia.org/wiki/Line_(geometry))

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Acknowledgments

None so far.

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Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas. Mathematical equations are prepared in MathType by Design Science, Inc. Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see <https://www.rbg.ca/> for more information.



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The equations of a line 

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